Solutions 11

Exercise 3.1

(a)

$$Z_k = \ln\left(\frac{f_1(Y_k)}{f_0(Y_k)}\right) = \ln\left(\frac{\lambda_1}{\lambda_0}\right) - Y_k(\lambda_1 - \lambda_0). \tag{1}$$

$$\begin{aligned} G_0(u) &= \int_0^\infty \lambda_1^u \lambda_0^{1-u} e^{-\lambda_1 y u} e^{-\lambda_0 y(1-u)} dy \\ &= \lambda_1^u \lambda_0^{1-u} \int_0^\infty e^{-y(\lambda_1 u + \lambda_0(1-u))} dy \\ &= \frac{\lambda_1^u \lambda_0^{1-u}}{\lambda_1 u + \lambda_0(1-u)} \end{aligned} \tag{2}$$

(b)

$$\ln\left(G_0(u)\right) = u \ln \lambda_1 + (1-u)\ln\lambda_0 - \ln\left[\lambda_1 u + \lambda_0(1-u)\right]. \tag{3}$$

$$\frac{\partial \ln G_0(u)}{\partial u} = \ln \lambda_1 - \ln \lambda_0 - \frac{\lambda_1 - \lambda_0}{\lambda_1 \ln u + \lambda_0(1-u)}. \tag{4}$$

Then, we can derive m_0 and m_1 as follows:

$$m_0 = \frac{\partial \ln\left(G_0(0)\right)}{\partial u} = \ln\left(\frac{\lambda_1}{\lambda_0}\right) - \left(\frac{\lambda_1}{\lambda_0} - 1\right). \tag{5}$$

$$m_1 = \frac{\partial \ln\left(G_0(1)\right)}{\partial u} = \ln\left(\frac{\lambda_1}{\lambda_0}\right) - \left(1 - \frac{\lambda_0}{\lambda_1}\right). \tag{6}$$

(c)

$$\begin{aligned} z &= \ln \lambda_1 - \ln \lambda_0 - \frac{\lambda_1 - \lambda_0}{\lambda_1 \ln u + \lambda_0(1-u)} \\ u^* &= \frac{1}{\ln\lambda_1 - \ln\lambda_0 - z} - \frac{\lambda_0}{\lambda_1 - \lambda_0}. \end{aligned} \tag{7}$$

Then,

$$\begin{aligned} I_0(z) &= \max_{u \in \mathbb{R}} \left(zu - \ln\left(G_0(u)\right)\right) \\ &= zu^* - \ln G_0(u^*). \end{aligned} \tag{8}$$

(d) Since $z = 0$,

$$u^* = \frac{1}{\ln\lambda_1 - \ln\lambda_0} - \frac{\lambda_0}{\lambda_1 - \lambda_0}. \tag{9}$$

Then,

$$I_0(0) = -\ln G_0\left(\frac{1}{\ln\lambda_1 - \ln\lambda_0} - \frac{\lambda_0}{\lambda_1 - \lambda_0}\right) = -1 + \frac{\lambda_0 \ln(\frac{\lambda_1}{\lambda_0})}{\lambda_1 - \lambda_0} + \ln\left(\frac{\lambda_1 - \lambda_0}{\lambda_0 \ln(\frac{\lambda_1}{\lambda_0})}\right). \quad (10)$$

It can decay by exponential rate.

Exercise 3.2

(a) According to the result of Example 3.2, we know

$$\ln G_0(u) = -\frac{u(1-u)}{2} d^2, \quad (11)$$

where $d^2 = (m_1 - m_0)^2/\sigma^2$.

(b)

$$\frac{\partial \ln\left(G_0(u)\right)}{\partial u} = \frac{(m_1 - m_0)^2}{\sigma^2}(u - \frac{1}{2}). \quad (12)$$

Then, we can derive the m_0 and m_1:

$$m_0 = -\frac{(m_1 - m_0)^2}{2\sigma^2}, \quad (13)$$

$$m_1 = \frac{(m_1 - m_0)^2}{2\sigma^2}. \quad (14)$$

(c)

$$I_0(z) = \frac{(z + d^2/2)^2}{2d^2} = \frac{z}{2} + \frac{(m_1 - m_0)^2}{8\sigma^2} + \frac{\sigma^2 z^2}{2(m_1 - m_0)^2}. \quad (15)$$

(d)

$$I_0(0) = \frac{(m_1 - m_0)^2}{8\sigma^2}. \quad (16)$$

When m_1 and m_0 exist more difference, the decay will be more fast.

Exercise 3.3

(a) According to the form of Example 3.2, we know that

$$\begin{cases} m_0 = m_1 = 0 \\ K_0 = \sigma_0^2 \\ K_1 = \sigma_1^2 \\ K = \frac{\sigma_0^2 \sigma_1^2}{u\sigma_0^2 + (1-u)\sigma_1^2}. \end{cases} \quad (17)$$

And

$$\ln\left(G_0(u)\right) = \frac{1}{2}\Big[\ln\left(|K|(u)\right) - u\ln(|K_1|) - (1-u)\ln(|K_0|) \\ - u(1-u)\triangle m^T\left[uK_0 + (1-u)K_1\right]^{=1}\Big], \quad (18)$$

We can prove that

$$\ln G_0(u) = \frac{1}{2}\left[-\ln\left(\frac{u}{\sigma_1^2}+\frac{1-u}{\sigma_0^2}\right) - u\ln(\sigma_1^2) - (1-u)\ln(\sigma_0^2)\right]. \tag{19}$$

(b)

$$\frac{\partial \ln(G_0(u))}{\partial u} = \frac{\sigma_1^2-\sigma_0^2}{2(u\sigma_0^2+(1-u)\sigma_1^2)} - \ln\frac{\sigma_1}{\sigma_0}. \tag{20}$$

$$m_0 = \frac{\partial \ln(G_0(0))}{\partial u} = \frac{\sigma_1^2-\sigma_0^2}{2\sigma_1^2} - \ln\frac{\sigma_1}{\sigma_0}. \tag{21}$$

$$m_1 = \frac{\partial \ln(G_0(1))}{\partial u} = \frac{\sigma_1^2-\sigma_0^2}{2\sigma_0^2} - \ln\frac{\sigma_1}{\sigma_0}. \tag{22}$$

(c)

$$z = \frac{\sigma_1^2-\sigma_0^2}{2(u\sigma_0^2+(1-u)\sigma_1^2)} - \ln\frac{\sigma_1}{\sigma_0}, \tag{23}$$

we can have

$$u^* = -\frac{1}{2(z+\ln\frac{\sigma_1}{\sigma_0})} - \frac{\sigma_1^2}{\sigma_0^2-\sigma_1^2}. \tag{24}$$

Therefore,

$$I_0(z) = zu^* - \ln G_0(u^*). \tag{25}$$

(d)

$$u^* = \frac{1}{\ln\frac{\sigma_0^2}{\sigma_1^2}} - \frac{\sigma_1^2}{\sigma_0^2-\sigma_1^2}. \tag{26}$$

Exercise 3.4

(a)

$$\begin{aligned} G_0(u) &= \int_{-w_0/2}^{w_0/2}\left(\frac{w_0}{w_1}\right)^u \frac{1}{w_0}dy \\ &= \left(\frac{w_0}{w_1}\right)^u. \end{aligned} \tag{27}$$

Since we need $\ln G_0(u) = 0$ as well when $u = 1$, then there is discontinuous at $u = 1$.

(b)

$$m_0 = \int \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right) f(y|H_0)dy = \ln\frac{w_0}{w_1}, \tag{28}$$

$$m_1 = \int \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right) f(y|H_1)dy = \frac{w_0}{w_1}\ln\frac{w_0}{w_1}. \tag{29}$$

(c) When $u \neq 1$:

$$z = \ln\frac{w_0}{w_1}. \tag{30}$$

$$I_0(z) = \max_{u\in\mathbb{R}} zu - u\ln\frac{w_0}{w_1} = 0. \tag{31}$$

When $u = 1$:

$$I_0(z) = m_1 - \ln\frac{w_0}{w_1} = \ln\frac{w_0}{w_1}\left(1 - \frac{w_0}{w_1}\right). \tag{32}$$

Increasing the number of observations will not change $\mathsf{P_F}$ and $\mathsf{P_M}$ except the case that $u = 1$.

Exercise 3.5

(a)

$$\begin{aligned} G_0(u) &= \int_0^{1/2} (4y)^u dy + \int_{1/2}^{1} 4^u(1-y)^u dy \\ &= \frac{2^u}{u+1} \end{aligned} \tag{33}$$

(b)

$$\frac{\partial \ln(G_0(u))}{\partial u} = \ln 2 - \frac{1}{u+1}. \tag{34}$$

$$m_0 = \frac{\partial \ln(G_0(0))}{\partial u} = \ln 2 - 1, \tag{35}$$

$$m_1 = \frac{\partial \ln(G_0(1))}{\partial u} = \ln 2 - \frac{1}{2}. \tag{36}$$

(c)

$$z = \ln 2 - \frac{1}{u+1}, \tag{37}$$

then

$$u^* = \frac{1}{\ln 2 - z} - 1. \tag{38}$$

$$I_0(z) = \ln\left(\frac{2}{\ln 2 - z}\right) - (1+z). \tag{39}$$

(d)

$$I_0(0) = \ln\left(\frac{2}{\ln 2}\right) - 1 = 0.0597. \tag{40}$$